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Coefficient Inequalities of a Comprehensive Subclass of Analytic Functions With Respect to Symmetric Points

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Abstract

We have introduced a comprehensive subclass of analytic functions with respect to $(j,\ k)$ - symmetric points. We have obtained the interesting coefficient bounds for the newly defined classes of functions. Further, we have extended the study using quantum calculus. Our main results have several applications, here we have presented only a few of them.

Keywords: analytic functions; starlike and convex functions; subordination; coefficient inequalities; q-calculus.

1 Introduction

Let Π be collection of analytic functions defined in $\Omega = \{\xi : |\xi| < 1\}$ and has an expansion

$$\chi(\xi) = \xi + \sum_{\ell=2}^{\infty} a_\ell \xi^\ell. \tag{1}$$

We denote $\mathcal{P}$ to be family of functions $p(\xi) \in \Pi$ with $p(0) = 1$ and whose real part is more than zero. Let $\mathcal{S}$ denote the class of functions $\chi \in \Pi$ which are univalent in Ω. The class $\mathcal{S}$ is not preserved under addition. However, the class is preserved under k-root transformation. It is well known that if $\chi(\xi)$ given by (1) is in $\mathcal{S}$, then $\left[\chi(\xi^k)\right]^{1/k}$, ($k$ is a positive integer) is also in $\mathcal{S}$. Now we present the formal definition of k-symmetric function.

Definition 1.1. [18, *pg. 18*] *Let k be a positive integer. A domain $\mathbb{D}$ is said to be k-fold symmetric if a rotation of $\mathbb{D}$ about the origin through an angle $2\pi/k$ carries $\mathbb{D}$ onto itself. A function is said to be k-symmetric in Ω, if for ξ in Ω*

$$\chi(\varepsilon\xi) = \varepsilon\chi(\xi),$$

where $\varepsilon = \exp(2\pi i/k)$.

Extending the notion of functions with respect to $k-$ symmetric points, Liczberski and Połubiński in [28] introduced the so called class of functions with respect to (j, k)-symmetric points. For every integer j, a function $\chi \in \Pi$ is said to be (j, k)-symmetrical if for each $\xi \in \Omega$

$$\chi(\varepsilon\xi) = \varepsilon^j\chi(\xi), \tag{2}$$

where $k \geq 2$ is a fixed integer, $j = 0, 1, 2, \ldots, k-1$ and $\varepsilon = \exp(2\pi i/k)$. We $\mathcal{F}_k^j$ to denote the class of functions satisfying (2). We observe that $\mathcal{F}_2^1$, $\mathcal{F}_2^0$ and $\mathcal{F}_k^1$ are well-known families of odd functions, even functions and k-symmetrical functions respectively. For every integer j, let $\chi_{j,\,k}(\xi)$ be defined by the following equality

$$\chi_{j,\,k}(\xi) = \frac{1}{k}\sum_{\nu=0}^{k-1} \frac{\chi(\varepsilon^\nu\xi)}{\varepsilon^{\nu j}}, \quad (\chi \in \Pi). \tag{3}$$

From (3), we see that $\chi_{j,\,k}(\xi)$ satisfies the linearity conditions.

Two well-known subclasses of Π are the so-called starlike functions of order γ and convex functions of order γ, which satisfies the differential inequality of the form

$$Re\left(\frac{\xi\chi'(\xi)}{\chi(\xi)}\right) > \gamma \quad \text{and} \quad Re\left(1 + \frac{z\chi''(\xi)}{\chi'(\xi)}\right) > \gamma, \qquad (z \in \Omega;\ 0 \leq \gamma < 1),$$

respectively. We let $\mathcal{S}^*(\gamma)$ and $\mathcal{C}_p(\gamma)$ to denote the respective class of starlike functions of order γ and convex functions of order γ. Extending the class $\mathcal{S}^*(\gamma)$, Sakaguchi [36] defined the class $\mathcal{S}_s^*(\gamma)$, the class of function starlike with respect to k-symmetric points. The function $\chi \in \Pi$ is said to be in $\mathcal{S}_s^*(\gamma)$ if it satisfies the analytic characterization

$$Re\left(\frac{\xi\chi'(\xi)}{\chi_k(\xi)}\right) > \gamma,$$

where $\chi_k(\xi) = \frac{1}{k}\sum_{\nu=0}^{k-1}\frac{\chi(\varepsilon^\nu\xi)}{\varepsilon^\nu}$, $\quad (\chi \in \Pi)$. Further, Sakaguchi established that classes of functions in $\mathcal{S}_s^*(\gamma)$ are univalent. For developments and study of various subclasses of analytic functions, refer to [18].

The main tools that are used to study the various geometrically defined subclasses of analytic functions are Hadamard product and subordination. For formal definition of Hadamard product and subordination, refer to Srivastava [43]. Here we let $*$ and $\prec$ to denote the Hadamard product and subordination respectively. Using subordination, Ma and Minda [29] introduced the following

$$\mathcal{S}^*(\psi) = \left\{\chi \in \Pi : \frac{\xi\chi^{'}(\xi)}{\chi(\xi)} \prec \psi\right\} \quad \text{and} \quad \mathcal{C}(\psi) = \left\{\chi \in \Pi : 1 + \frac{\xi\chi''(\xi)}{\chi'(\xi)} \prec \psi\right\},$$

where $\psi \in \mathcal{P}$ with $\psi^{'}(0) > 0$ maps Ω onto a region starlike with respect to 1 and symmetric with respect to real axis.

For arbitrary fixed numbers $G,\, H,\, -1 < G \leq 1,\, -1 \leq H < G$, we denote by $\mathcal{P}(G,\, H)$ the family of functions $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots$ analytic in the unit disc and $p(\xi) \in \mathcal{P}(G,\, H)$ if and only if

$$p(\xi) = \frac{1 + Gw(\xi)}{1 + Hw(\xi)},$$

where $w(\xi)$ is the Schwarz function. The functions in $\mathcal{P}(G,\, H)$ are popularly known of Janowski functions (see [20]).

For $\chi \in \Pi$, the q-difference operator (see [8, 10]) for $\chi \in \Pi$ is defined by $(0 < q < 1)$,

$$\mathfrak{D}_q\chi(\xi) = \begin{cases} \chi^{'}(0), & \text{if } \xi = 0, \\ \frac{\chi(\xi) - \chi(q\xi)}{(1-q)\xi}, & \text{if } \xi \neq 0. \end{cases} \tag{4}$$

From (4), we can easily see that $\mathfrak{D}_q\chi(\xi) = 1 + \sum_{\ell=2}^{\infty}[\ell]_q a_\ell \xi^{\ell-1},\ (\xi \neq 0)$, where the q-integer number $[\ell]_q$ is defined by

$$[\ell]_q = \frac{1 - q^{\ell}}{1 - q} \tag{5}$$

and note that $\lim_{q\to 1^-} \mathfrak{D}_q\chi(\xi) = \chi'(\xi)$. We denote $\mathfrak{D}_q^2\chi(\xi) = \mathfrak{D}_q\left[\mathfrak{D}_q\chi(\xi)\right]$. It should be noted that everything in classical calculus cannot be generalized to quantum calculus, notably the chain rule needs adaptation. So recently there is renewed interest in all the fields of research to replace the classical derivative with a quantum derivative, refer to [3, 7, 21, 31, 37, 46, 47] for recent developments involving q-calculus. The duality theory of *Quantum Calculus* and *Univalent Function Theory* was introduced by Srivastava [42]. For recent developments and applications of quantum calculus pertaining to the study of various subclasses, refer to the recent article of Srivastava (2020) [43] and references provided therein. For general theory and analysis involving theory of special functions, refer to [1, 2, 11, 35]. We denote

$$([\ell]_q)_t = [\ell]_q[\ell+1]_q[\ell+2]_q \ldots [\ell+t-1]_q.$$

By using Hadamard product, *q-Calson-Shaffer operator* (see [40]) is defined by

$$\mathcal{L}_q(b,\, c)\chi(\xi) = \xi + \sum_{\ell=2}^{\infty} \frac{([b]_q)_{\ell-1}}{([c]_q)_{\ell-1}} a_\ell \xi^{\ell} = \xi + \sum_{\ell=2}^{\infty} \Upsilon_\ell a_\ell \xi^{\ell}. \tag{6}$$

We will denote $\Phi_\ell = \lim_{q\to 1^-} \Upsilon_\ell = \frac{(b)_{\ell-1}}{(c)_{\ell-1}}$. Notice that $\mathfrak{D}_q(\mathcal{L}_q(b,\, c)\chi(\xi)) = 1 + \sum_{\ell=2}^{\infty} \Upsilon_\ell [\ell]_q\, a_\ell \xi^{\ell-1}$. Further, we observe that if $\lim_{q\to 1^-} \mathcal{L}_q(b,\, c)\chi(\xi) = L(b,\, c)\chi(\xi)$, where $L(b,\, c)\chi(\xi)$ is the well-known *Carlson-Shaffer operator* [13]. Further $\mathcal{L}_q(\mu+1,\, 1)\chi(\xi) = \mathcal{R}_\mu^q\chi(\xi)\,(\mu > 0)$, where $\mathcal{R}_\mu^q\chi(\xi)$

is the *Ruscheweyh q-derivative operator*. The q-analogues of the various other differential operators, refer to [19, 22]. It is easily verified from (6) that

$$q^{b-1}\xi\left(\mathfrak{D}_q\mathcal{L}_q(b,\,c)\chi(\xi)\right)=[b]_q\mathcal{L}_q(b+1,\,c)\chi(\xi)-[b-1]_q\mathcal{L}_q(b,\,c)\chi(\xi).$$

Linear operators defined using special functions have been very helpful in consolidating the study of various subclasses of analytic functions and also have been very useful in extracting the information of the various properties of analytic functions. Here using a well-known Carlson-Shaffer operator, we define a very comprehensive subclasses of analytic functions with respect to symmetric points. Coefficient inequalities are the main results of this paper. When it pertained to applications of our main results, we have focussed on the results involving conic regions.

We assume that $k\in\mathbb{N}$, $\varepsilon=\exp(2\pi i/k)$ and

$$\mathcal{L}^q_{j,\,k}(\xi)=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{-\nu j}\left[\mathcal{L}_q(b,\,c)\chi(\varepsilon^\nu\xi)\right]=\xi+\cdots, \tag{7}$$

From (7), we can get

$$\mathcal{L}^q_{j,\,k}(\xi)=\sum_{\ell=1}^{\infty}a_\ell\Upsilon_\ell\Gamma_{\ell,j}\xi^\ell,\quad (a_1=\Upsilon_1=1),\quad \Gamma_{\ell,j}=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{(\ell-j)\nu} \tag{8}$$

and we let

$$L_{j,\,k}(\xi)=\lim_{q\to1^-}\mathcal{L}^q_{j,\,k}(\xi)=\sum_{\ell=1}^{\infty}a_\ell\Phi_\ell\Gamma_{\ell,j}\xi^\ell. \tag{9}$$

Extending the notion introduced by Sakaguchi in [36], several subclasses of analytic functions with respect to other points were introduced and studied by various authors (see [5, 23, 32, 38, 39, 44]).

Definition 1.2. *For $0\le\delta\le\lambda\le1$ and $-1\le H<G\le1$, the function $\chi\in\Pi$ is said to be in $\mathcal{C}_{(j,k)}(b,c;\lambda,\delta;\psi;\,G,H)$ if it satisfies the condition*

$$\frac{(1-\lambda+\delta)L(b,\,c)\chi(\xi)+(\lambda-\delta)\xi L(b,\,c)\chi'(\xi)+\lambda\delta\xi^2L(b,\,c)\chi''(\xi)}{L_{j,\,k}(\xi)}\prec\frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)}, \tag{10}$$

where $L_{j,\,k}(\xi)$ is defined as in (9) and $\psi\in\mathcal{P}$ which has a power series expansion of the form

$$\psi(\xi)=1+L_1\xi+L_2\xi^2+L_3\xi^3+\cdots,\ \xi\in\Omega,\ L_1>0. \tag{11}$$

Definition 1.3. *For $0\le\delta\le\lambda\le1$ and $-1\le H<G\le1$, the function $\chi\in\Pi$ is said to be in $\mathcal{K}_{(j,k)}(b,c;\lambda,\delta;\psi;\,G,H)$ if it satisfies the condition*

$$\frac{L(b,\,c)\chi'(\xi)+(\lambda-\delta+2\lambda\delta)\xi L(b,\,c)\chi''(\xi)+\lambda\delta\xi^2L(b,\,c)\chi'''(\xi)}{(L_{j,\,k}(\xi))'}\prec\frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)}, \tag{12}$$

where $L_{j,\,k}(\xi)$ and ψ are defined as in (9) and (11) respectively.

Remark 1.1. *The classes $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ and $\mathcal{K}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ were mainly motivated by the comprehensive class defined by Bulut (2020) [12]. Here we list few special cases of the class.*

1. *If we let $b = c$, $\lambda = 1$, $\delta = 0$ and $\psi(\xi) = 1+\xi/1-\xi$ in Definition 1.2, then the class $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ reduces to the class $\mathcal{S}^{(j,\,k)}[F,\ G]$ defined and studied by Al Sarari et al. [5, Definition 5].*
2. *If we let $b = c$, $\lambda = 1$, $\delta = 0$, $j = 1$ and $\psi(\xi) = 1 + \xi/1 - \xi$ in Definition 1.2, then the class $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ reduces to the class $\mathcal{S}_k^{(k)}[F,\ G]$ defined and studied by Kwon, and Sim [27].*

Further, the classes recently introduced and studied by Sarari et al. (2019) in [6] is closely related to the classes $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ and $\mathcal{K}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ defined by us here.

Definition 1.4. *For $0 \le \delta \le \lambda \le 1$ and $-1 \le H < G \le 1$, the function $\chi \in \Pi$ is said to be in the class $\mathcal{QC}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ if it satisfies the subordination condition*

$$\begin{aligned}\frac{(1-\lambda+\delta)\mathcal{L}_q(b,\ c)\chi(\xi) + (\lambda-\delta)\xi\mathfrak{D}_q\mathcal{L}_q(b,\ c)\chi(\xi) + \lambda\delta\xi^2\mathfrak{D}_q^2\mathcal{L}_q(b,\ c)\chi(\xi)}{\mathcal{L}^q_{j,\,k}(\xi)}\\ \prec \frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)},\end{aligned} \tag{13}$$

where $\mathcal{L}^q_{j,\,k}(\xi)$ and ψ are defined as in (8) and (11) respectively.

Definition 1.5. *For $0 \le \delta \le \lambda \le 1$ and $-1 \le H < G \le 1$, the function $\chi \in \Pi$ is said to be in the class $\mathcal{QK}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ if it satisfies the subordination condition*

$$\begin{aligned}\frac{\mathfrak{D}_q\mathcal{L}_q(b,\ c)\chi(\xi) + (\lambda-\delta+2\lambda\delta)\xi\mathfrak{D}_q^2\mathcal{L}_q(b,\ c)\chi(\xi) + \lambda\delta\xi^2\mathfrak{D}_q^3\mathcal{L}_q(b,\ c)\chi(\xi)}{\mathfrak{D}_q\mathcal{L}^q_{j,\,k}(\xi)}\\ \prec \frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)},\end{aligned} \tag{14}$$

where $\mathcal{L}^q_{j,\,k}(\xi)$ and ψ are defined as in (8) and (11) respectively.

Remark 1.2. *If we let $\lambda = 1$, $\delta = 0$, $b = c$ and $\psi(\xi) = \frac{1+\xi}{1-\xi}$, then the class $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\ G, H)$ reduces to the class $\mathcal{S}^{(j,\,k)}(G,\ H)$. The class $\mathcal{S}^{(j,\,k)}(G,\ H)$ was introduced by Sarari et al. (2016) [5, Definition 5].*

2 Coefficient Inequalities

In order to obtain our main results, we need the following Lemma recently obtained by Karthikeyan et al. [25].

Lemma 2.1. [25] *Let the function $\dfrac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)}$ be convex in $\mathbb{U}$ where the function ψ is defined as in (11). If $p(\xi) = 1 + \sum\limits_{\ell=1}^{\infty} p_\ell\xi^\ell$ is analytic in $\mathbb{U}$ and satisfies the subordination condition*

$$p(\xi) \prec \frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)}, \tag{15}$$

then

$$|p_\ell| \le \frac{|L_1|(G-H)}{2},\ \ell \ge 1. \tag{16}$$

As evident from the area theorem, coefficient inequalities play an central role in studying various properties of subclasses of analytic functions. To reemphasize, note that

$$\Phi_\ell = \frac{(b)_{\ell-1}}{(c)_{\ell-1}} \quad \text{and} \quad \Upsilon_\ell = \frac{([b]_q)_{\ell-1}}{([c]_q)_{\ell-1}}.$$

Here we present the coefficient inequality of $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\, G, H)$.

Theorem 2.1. *Let Φ_ℓ's be real and $-1 \leq H < 0$. If $\chi \in \mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\, G, H)$, then for $\ell \geq 2$,*

$$|a_\ell| \leq \frac{1}{\Phi_\ell} \prod_{t=1}^{\ell-1} \frac{|(G-H)L_1\Gamma_{t,j} - 2\,[1 + (\lambda-\delta)(t-1) + \lambda\delta t(t-1) - \Gamma_{t,j}]\, H|}{2\,|1 + t(\lambda-\delta) + \lambda\delta t(t+1) - \Gamma_{t+1,j}|}. \tag{17}$$

Proof. By the definition of $\mathcal{C}_{(j,k)}(b, c; \lambda, \delta; \psi;\, G, H)$, we have

$$\frac{(1-\lambda+\delta)L(b,\, c)\chi(\xi) + (\lambda-\delta)\xi L(b,\, c)\chi'(\xi) + \lambda\delta\xi^2 L(b,\, c)\chi''(\xi)}{L_{j,\, k}(\xi)} = p(\xi), \tag{18}$$

where $p(\xi) \in \mathcal{P}$ is subordinate to $p(\xi) \prec \frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)}$.

Equivalently, (18) can be written as

$$\begin{aligned}
(1-\Gamma_{1,j})\xi + &\sum_{\ell=2}^{\infty}[1 - \Gamma_{\ell,j} + (\lambda-\delta)(\ell-1) + \lambda\delta\ell(\ell-1)]\Phi_\ell a_\ell \xi^\ell \\
&= \left(\sum_{\ell=1}^{\infty} p_\ell \xi^\ell\right)\left(\sum_{\ell=1}^{\infty} \Gamma_{\ell,j}\Phi_\ell a_\ell \xi^\ell\right) \qquad (a_1 = \Phi_1 = \Gamma_{1,j} = 1)\,.
\end{aligned}$$

Equating the coefficient of ξ^ℓ on both sides

$$\begin{aligned}
[1 - \Gamma_{\ell,j} + (\lambda-\delta)(\ell-1) + \lambda\delta\ell(\ell-1)]\Phi_\ell a_\ell &= [\Gamma_{\ell-1,j}\Phi_{\ell-1}a_{\ell-1}p_1 + \cdots + p_{\ell-1}\Gamma_{1,j}\Phi_1 a_1] \\
&= \sum_{t=1}^{\ell-1} |p_t\, \Gamma_{t,j}\, \Phi_{t,j}\, a_t| \leq \sum_{t=1}^{\ell-1} |p_t\, \Gamma_{t,j}|\, \Phi_t\, |a_t|.
\end{aligned}$$

From Lemma 2.1, we have $|p_\ell| \leq \frac{|L_1|(G-H)}{2}$, $\ell \geq 1$. On computation we have

$$|a_\ell| \leq \frac{(G-H)|L_1| \sum_{t=1}^{\ell-1} \Phi_t |\Gamma_{t,j}\, a_t|}{2|1 + (\lambda-\delta)(\ell-1) + \lambda\delta\ell(\ell-1)] - \Gamma_{\ell,j}|\Phi_\ell}. \tag{19}$$

Let $\ell = 2$ in (19), then

$$|a_2| \leq \frac{(G-H)\,|L_1\,\Gamma_{1,j}|}{2|1 + (\lambda-\delta) + 2\lambda\delta] - \Gamma_{2,j}|\Phi_2}. \tag{20}$$

By letting $\ell = 2$ in (17), we get

$$\begin{aligned}
|a_2| &\leq \frac{1}{\Phi_2} \prod_{t=1}^{2-1} \frac{|(G-H)L_1\Gamma_{t,j} - 2\,[1 + (\lambda-\delta)(t-1) + \lambda\delta t(t-1) - \Gamma_{t,j}]\, H|}{2\,|1 + t(\lambda-\delta) + \lambda\delta t(t+1) - \Gamma_{t+1,j}|} \\
&= \frac{1}{\Phi_2} \frac{|(G-H)L_1\Gamma_{1,j} - 2\,[1 - \Gamma_{1,j}]\, H|}{2\,|1 + (\lambda-\delta) + 2\lambda\delta - \Gamma_{2,j}|} = \frac{1}{\Phi_2} \frac{(G-H)\,|L_1\,\Gamma_{1,j}|}{2\,|1 + (\lambda-\delta) + 2\lambda\delta - \Gamma_{2,j}|}.
\end{aligned} \tag{21}$$

From (20) and (21), we find that hypothesis is correct for $\ell = 2$. Similarly if we let $\ell = 3$ in (19), we have

$$\begin{aligned}
|a_3| &\leq \frac{(G-H)|L_1|}{2\,|1+2(\lambda-\delta)+6\lambda\delta-\Gamma_{3,j}|\,\Phi_3}\left[|\Gamma_{1,j}|+|\Gamma_{2,j}|\,\Phi_2|a_2|\right] \\
&\leq \frac{(G-H)\,|L_1\,\Gamma_{1,j}|}{2\,|1+2(\lambda-\delta)+6\lambda\delta-\Gamma_{3,j}|\,\Phi_3}\left[1+\frac{|L_1|(G-H)\,|\Gamma_{2,j}|}{2\,|1+(\lambda-\delta)+2\lambda\delta-\Gamma_{2,j}|}\right].
\end{aligned}$$

If we let $\ell = 3$, in (17), we have

$$\begin{aligned}
|a_3| &\leq \frac{1}{\Phi_3}\left[\frac{(G-H)\,|L_1\,\Gamma_{1,j}|}{2\,|1+(\lambda-\delta)+2\lambda\delta-\Gamma_{2,j}|}\times\frac{|(G-H)L_1\Gamma_{2,j}-2[1+(\lambda-\delta)+2\lambda\delta-\Gamma_{2,j}]H|}{2\,|1+2(\lambda-\delta)+6\lambda\delta-\Gamma_{3,j}|}\right] \\
&\leq \frac{1}{\Phi_3}\left[\frac{(G-H)\,|L_1\,\Gamma_{1,j}|}{2\,|1+2(\lambda-\delta)+6\lambda\delta-\Gamma_{3,j}|}\times\frac{(G-H)\,|L_1\Gamma_{2,j}|+2[1+(\lambda-\delta)+2\lambda\delta-\Gamma_{2,j}]|H|}{2[1+(\lambda-\delta)+2\lambda\delta-\Gamma_{2,j}]}\right] \\
&\leq \frac{(G-H)\,|L_1\,\Gamma_{1,j}|}{2\,|1+2(\lambda-\delta)+6\lambda\delta-\Gamma_{3,j}|\,\Phi_3}\left[1+\frac{|L_1|(G-H)\,|\Gamma_{2,j}|}{2\,|1+(\lambda-\delta)+2\lambda\delta-\Gamma_{2,j}|}\right].
\end{aligned}$$

Hence, the hypothesis is correct for $\ell = 3$. Assume that (17) is valid for $\ell = 2, 3, \ldots r$. Applying triangle inequality in (17), we obtain

$$|a_r| \leq \frac{1}{\Phi_r}\prod_{t=1}^{r-1}\frac{(G-H)\,|L_1\Gamma_{t,j}|+2\,|1+(\lambda-\delta)(t-1)+\lambda\delta t(t-1)-\Gamma_{t,j}|}{2\,|1+t(\lambda-\delta)+\lambda\delta t(t+1)-\Gamma_{t+1,j}|}.$$

By induction hypothesis, we have

$$\begin{aligned}
&\frac{(G-H)|L_1\,\Gamma_{r,j}|}{2\,|1+(\lambda-\delta)(r-1)+\lambda\delta r(r-1)]-\Gamma_{r,j}|}\left[\sum_{t=1}^{r-1}|\Gamma_{t,j}\,\Phi_t\,a_t|\right] \\
\leq\ &\frac{1}{\Phi_r}\prod_{t=1}^{r-1}\frac{(G-H)\,|L_1\Gamma_{t,j}|+2\,|1+(\lambda-\delta)(t-1)+\lambda\delta t(t-1)-\Gamma_{t,j}|}{2\,|1+t(\lambda-\delta)+\lambda\delta t(t+1)-\Gamma_{t+1,j}|}.
\end{aligned}$$

From the above inequality, we have

$$\begin{aligned}
&\frac{1}{\Phi_{r+1}}\prod_{t=1}^{r}\frac{(G-H)\,|L_1\Gamma_{t,j}|+2\,|1+(\lambda-\delta)(t-1)+\lambda\delta t(t-1)-\Gamma_{t,j}|}{2\,|1+t(\lambda-\delta)+\lambda\delta t(t+1)-\Gamma_{t+1,j}|} \\
&\qquad\geq \frac{(G-H)\,|L_1|}{2\,|1+(\lambda-\delta)(r-1)+\lambda\delta r(r-1)]-\Gamma_{r,j}|}\times \\
&\frac{(G-H)\,|L_1\,\Gamma_{r,j}|+2\,|1+(\lambda-\delta)(r-1)+\lambda\delta r(r-1)-\Gamma_{r,j}|}{2\,|1+r(\lambda-\delta)+\lambda\delta r(r+1)-\Gamma_{r+1,j}|}\left[\sum_{t=1}^{r-1}\Phi_t\,|\Gamma_{t,j}\,a_t|\right]
\end{aligned}$$

$$= \frac{(G-H)\,|L_1|}{2\,|1+r(\lambda-\delta)+\lambda\delta r(r+1)-\Gamma_{r+1,j}|}\times$$
$$\left[\frac{(G-H)\,|L_1\,\Gamma_{r,j}|}{2[1+(\lambda-\delta)(r-1)+\lambda\delta r(r-1)]-\Gamma_{r,j}]}+1\right]\left[\sum_{t=1}^{r-1}\Phi_t\,|\Gamma_{t,j}\,a_t|\right]$$
$$= \frac{(G-H)\,|L_1|}{2\,|1+r(\lambda-\delta)+\lambda\delta r(r+1)-\Gamma_{r+1,j}|}\times$$
$$\left[\frac{(G-H)\,|L_1\,\Gamma_{r,j}|}{2[1+(\lambda-\delta)(r-1)+\lambda\delta r(r-1)]-\Gamma_{r,j}]}\sum_{t=1}^{r-1}\Phi_t\,|\Gamma_{t,j}\,a_t|+\sum_{t=1}^{r-1}\Phi_t\,|\Gamma_{t,j}\,a_t|\right]$$
$$= \frac{(G-H)\,|L_1|}{2\,|1+r(\lambda-\delta)+\lambda\delta r(r+1)-\Gamma_{r+1,j}|}\left[\sum_{t=1}^{r}\Phi_t\,|\Gamma_{t,j}\,a_t|\right],$$

implies that inequality (17) is true for $\ell = r+1$. Hence the proof of the Theorem. □

Remark 2.1. *The presence of Carlson-Shaffer in the result provides more versatility to the result obtained in Theorem 2.1. Apart from including [5, Theorem 2] as a special case, Theorem 2.1 helps us to obtain the coefficient inequalities of various subclasses of convex functions with respect to $(j,\ k)$-symmetric points.*

We will now obtain the coefficient inequalities of the function class $\mathcal{K}_{(j,k)}(b,c;\lambda,\delta;\psi;\ G,H)$

Theorem 2.2. *Let Φ_ℓ's be real and $-1 \le H < 0$. If $\chi \in \mathcal{K}_{(j,k)}(b,c;\lambda,\delta;\psi;\ G,H)$, then for $\ell \ge 2$,*

$$|a_\ell| \le \frac{1}{\Phi_\ell}\prod_{t=1}^{\ell-1}\frac{|(G-H)tL_1\Gamma_{t,j}-2(t-1)\,[1+(\lambda-\delta)(t-1)+\lambda\delta t(t-1)-\Gamma_{t,j}]\,H|}{2t\,|1+t(\lambda-\delta)+\lambda\delta t(t+1)-\Gamma_{t+1,j}|}. \tag{22}$$

Proof. By the definition of $\mathcal{K}_{(j,k)}(b,c;\lambda,\delta;\psi;\ G,H)$ we have

$$\frac{L(b,\ c)\chi'(\xi)+(\lambda-\delta+2\lambda\delta)\xi L(b,\ c)\chi''(\xi)+\lambda\delta\xi^2L(b,\ c)\chi'''(\xi)}{(L_{j,\,k}(\xi))'} = p(\xi) \tag{23}$$

where $p(\xi)\in\mathcal{P}$ is subordinate to $p(\xi) \prec \frac{(G+1)\psi(\xi)-(G-1)}{(H+1)\psi(\xi)-(H-1)}$. Equivalently, (23) can be written as

$$1+\sum_{\ell=2}^{\infty}\ell a_\ell\Phi_\ell\xi^{\ell-1}[1+(\lambda+\delta+2\lambda\delta)(\ell-1)+\lambda\delta(\ell-1)(\ell-2)] = \left(1+\sum_{\ell=1}^{\infty}p_\ell\xi^\ell\right)\left(\sum_{\ell=1}^{\infty}\ell a_\ell\Gamma_{\ell,j}\xi^{\ell-1}\right).$$

By equating the coefficient of $\xi^{\ell-1}$ in the above equation and on computation we get ,

$$|a_\ell| \le \frac{|L_1|(G-H)\sum_{t=1}^{\ell-1}t\Phi_t\Gamma_{t,j}|a_t|}{2\ell\,|1+(\lambda+\delta+2\lambda\delta)(\ell-1)+\lambda\delta(\ell-1)(\ell-2)-\Gamma_{\ell,j}|}.$$

Now retracing the steps as in Theorem 2.1, we obtain the assertion of the Theorem. □

Keeping with the recent trend of studying the class of functions involving quantum calculus, we will now state the coefficient inequalities for the classes $\mathcal{QC}_{(j,k)}(b,c;\lambda,\delta;\psi;\ G,H)$ and $\mathcal{QK}_{(j,k)}(b,c;\lambda,\delta;\psi;\ G,H)$. Since the results did not involve quantum calculus chain rule or logarithmic differentiation which indeed requires some adaptation, here we choose to omit the details of the proof.

Theorem 2.3. *Let Υ_ℓ's be real and $-1 \leq H < 0$.. If $\chi \in \mathcal{QC}_{(j,k)}(b, c; \lambda, \delta; \psi; G, H)$, then for $\ell \geq 2$,*

$$|a_\ell| \leq \frac{1}{\Upsilon_\ell} \prod_{t=1}^{\ell-1} \frac{|(G-H)L_1\Gamma_{t,j} - 2\left[1 + (\lambda-\delta)([t-1]_q) + \lambda\delta[t]_q[t-1]_q - \Gamma_{t,j}\right] H|}{2\left|1 + [t]_q(\lambda-\delta) + \lambda\delta[t]_q[t+1]_q - \Gamma_{t+1,j}\right|}. \tag{24}$$

Theorem 2.4. *Let Φ_ℓ's be real and $-1 \leq H < 0$. If $\chi \in \mathcal{QK}_{(j,k)}(b, c; \lambda, \delta; \psi; G, H)$, then for $\ell \geq 2$,*

$$|a_\ell| \leq \frac{1}{\Upsilon_\ell} \prod_{t=1}^{\ell-1} \frac{\left| \begin{matrix} (G-H)[t]_q L_1 \Gamma_{t,j} \\ -2[t-1]_q[1+(\lambda-\delta)[t-1]_q + \lambda\delta[t]_q[t-1]_q - \Gamma_{t,j}]H \end{matrix} \right|}{2t\left|1 + t(\lambda-\delta) + \lambda\delta t(t+1) - \Gamma_{t+1,j}\right|}. \tag{25}$$

If we let $\lambda = 1$, $\delta = 0$, $b = c$ and $\psi(\xi) = \frac{1+\xi}{1-\xi}$ in Theorem 2.1, then we get the following result.

Corollary 2.1. [5, *Theorem 2*] *If $\chi \in \mathcal{S}^{(j,\,k)}(G,\,H)$, then for $\ell \geq 2$, $-1 \leq H < G \leq 1$,*

$$|a_\ell| \leq \prod_{t=1}^{\ell-1} \frac{|\Gamma_{t,j}|\,[(G-H)-1]+t}{|t+1-\Gamma_{t+1,j}|}.$$

If we let $\delta = 0$ and $\lambda = 1$ in Theorem 2.1, we have

Corollary 2.2. *If $\chi \in \Pi$ satisfies the condition*

$$\frac{\xi L(b,\,c)\chi'(\xi)}{L_{j,\,k}(\xi)} \prec \frac{(G+1)\psi(\xi) - (G-1)}{(H+1)\psi(\xi) - (H-1)},$$

then for $\ell \geq 2$, $-1 \leq H < G \leq 1$,

$$|a_\ell| \leq \frac{1}{\Phi_\ell} \prod_{t=1}^{\ell-1} \frac{|(G-H)L_1\Gamma_{t,j} - 2\left[t - \Gamma_{t,j}\right] H|}{2\left|(t+1) - \Gamma_{t+1,j}\right|}.$$

2.1 Applications to Petal Shaped Region and Leaf-Like Region

Sokół [41] and Raina & Sokół [33] studied a class of functions starlike with respect to Bernoulli lemniscate and lune respectively, by restricting the ψ to a specific conic region in the definition of $\mathcal{S}^*(\psi)$. For studies related to conic region, refer to [9, 15, 14, 16, 17, 26, 30, 34] and references provided therein. Ullah et al. in [45] defined a class of functions subordinate to

$$\psi(\xi) = 1 + \tanh \xi, \quad (\xi \in \Omega).$$

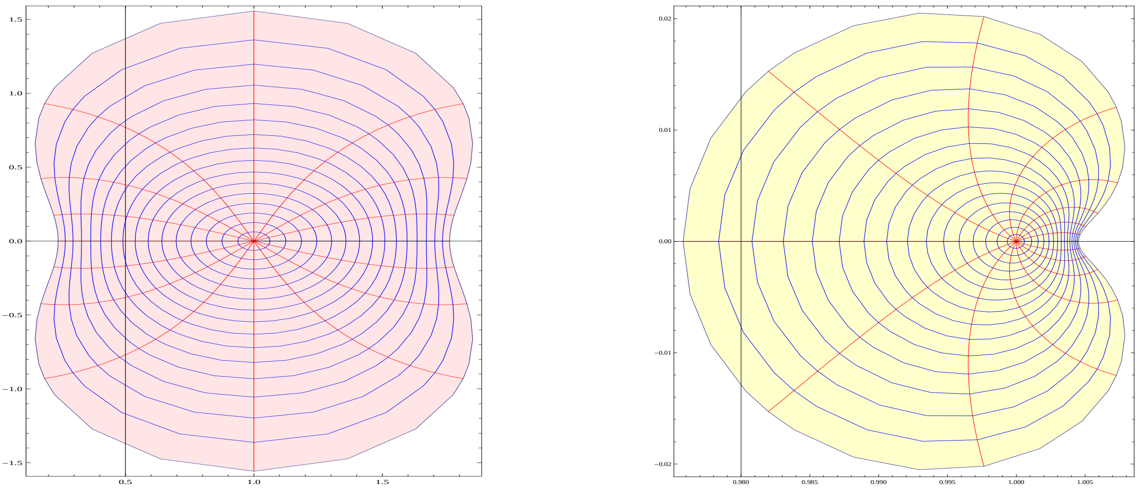
The function $\psi(\xi) = 1 + \tanh \xi$ maps the unit disc onto interior of the ellipsoid-like region which is symmetric about the real axis. (see Figure 1(a)). In Figure 1, we have shown that an ellipsoid-like region (see 1(a)) becomes a cardioid with cusp on the right hand side on the impact of Janowski function. The impact of Janowski function on various conic region has been studied by Karthikeyan et al. [24]. The function $1 + \tanh \xi$ has a Maclaurin series of the form

$$1 + \tanh \xi = 1 + \xi - \frac{\xi^3}{3} + \frac{2\xi^5}{15} - \frac{17\xi^7}{315} + \frac{62\xi^9}{2835} - \frac{1382\xi^{11}}{155925} + \frac{21844\xi^{13}}{6081075} - \frac{929569\xi^{15}}{638512875} + O[\xi]^{17}. \tag{26}$$

Ahuja et al. [4], defined a class of starlike functions subordinate to

$$\begin{aligned}\psi(\xi) &= 1 + \frac{\xi}{k}\left(\frac{k+\xi}{k-\xi}\right), \qquad (k = 1+\sqrt{2}),\\ &= 1 + \frac{\xi}{1+\sqrt{2}} + \frac{2\xi^2}{\left(1+\sqrt{2}\right)^2} + \cdots + \frac{2\xi^n}{\left(1+\sqrt{2}\right)^n} + O[\xi]^{n+1}.\end{aligned}$$

We can easily see that the function has a normalization $\psi(0) = 1$, $\mathrm{Re}\,[\psi(\xi)] > 0$ and maps unit disc on to the cardioid with cusp on the left hand side (also see Karthikeyan et al. [24]).



(a) (b)

Figure 1: (**a**) Mapping of Ω under $\Phi(\xi) = 1 + \tanh\xi$ (**b**)Impact of Janowski function on $\Phi(\xi) = 1 + \tanh\xi$, if $G = 0.82$ and $H = 0.8$.

Corollary 2.3. *Let* $\left|\dfrac{H-1}{H+1}\right| < 1$ *for all* $\xi \in \Omega$ *and* Φ_ℓ*'s be real. If* $\chi \in \Pi$ *satisfies the condition*

$$\frac{\xi\chi'(\xi)}{f_{j,\,k}(\xi)} \prec \frac{(G+1)[1+\xi-\xi^3/3]-(G-1)}{(H+1)[1+\xi-\xi^3/3]-(H-1)},$$

then for $\ell \geq 2$*,*

$$|a_\ell| \leq \frac{1}{\Phi_\ell}\prod_{t=1}^{\ell-1}\frac{|(G-H)\Gamma_{t,j} - 2\left[t-\Gamma_{t,j}\right]H|}{2\left[1+t-\Gamma_{t+1,j}\right]}.$$

Proof. If we let $\lambda = 1$, $\delta - 0$, $b = c$ and $\psi(\xi) = 1 + \tanh\xi$, in Definition 1.2, then we find from (26) $L_1 = 1$ and $L_2 = -\frac{1}{3}$. By substituting the appropriate parameter values in Theorem 2.1, we can establish the assertion of the Corollary. □

If we let $\lambda = 1$, $\delta = 0$, $b = c$ and $\psi(\xi) = 1 + \frac{\xi}{k}\left(\frac{k+\xi}{k-\xi}\right)$, in Theorem 2.1, we get the following result.

Corollary 2.4. *Let* $\left|\dfrac{H-1}{H+1}\right| < 1$ *for all* $\xi \in \Omega$ *and* Φ_ℓ*'s be real. If* $\chi \in \Pi$ *satisfies the condition*

$$\frac{\xi\chi'(\xi)}{f_{j,\,k}(\xi)} \prec \frac{(G+1)\left[1+\frac{\xi}{k}\left(\frac{k+\xi}{k-\xi}\right)\right]-(G-1)}{(H+1)\left[1+\frac{\xi}{k}\left(\frac{k+\xi}{k-\xi}\right)\right]-(H-1)},$$

then for $\ell \geq 2$*,*

$$|a_\ell| \leq \frac{1}{\Phi_\ell} \prod_{t=1}^{\ell-1} \frac{\left|(G-H)(\sqrt{2}-1)\Gamma_{t,j} - 2\left[t-\Gamma_{t,j}\right]H\right|}{2\left[1+t-\Gamma_{t+1,j}\right]}.$$

3 Conclusions

By defining a new comprehensive subclass of analytic functions, we were able to unify and generalize the various study of analytic functions with respect to $(j,\ k)$-symmetric points. Since very few study has been conducted on analytic functions with respect to $(j,\ k)$-symmetric points, only few special cases could be discussed. Further, by replacing the ordinary differentiation with quantum differentiation we have attempted at the discretization of some of the well-known results.

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